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Cosmological Model with Heat Flow

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An earlier work on a generalized Robertson–Walker–Friedmann metric with radial heat flow is extended to any arbitrary motion.

In an earlier work (Bokhari, n.d.) we considered a generalization of the space-time metric due to Glass (1979) and Bergmann (1975, 1981) to write

$$ds^2 = A^2(t, r)\,dt^2 - B^2(t, r)[(1 - kr^2/R^2)^{-1}\,dr^2 + r^2\,d\Omega^2] \tag{1}$$

where k corresponds to the three Friedmann cosmological models when $A = 1$ in equation (1) and

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\,d\phi^2 \tag{2}$$

The stress-energy-momentum tensor associated with the above metric is

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab} + q^a u^b + q^b u^a \tag{3}$$

where ρ, p, u^a, and q^a, respectively, represent mass-energy density, isotropic pressure, unit timelike 4-velocity vector defined by $u^a = A^{-1}\delta_0^a$, and the radial component of heat flow. We write the Einstein field equations in the form

$$\mathrm{X}_{ab}: \quad R_{ab} = 8\pi(T_{ab} - \tfrac{1}{2}g_{ab}T) \tag{4}$$

where gravitational units have been used to write χ. In the earlier work we had assumed $\ddot{B} = 0$. However, we now consider any arbitrary motion. Thus,

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$\ddot{B}\neq 0$. In this case, solving equations (4) for X_{01} yields the radial component of heat flow

$$q^a = 2(1-kr^2/R^2)/\chi B^2[\dot{B}/AB]'\delta_1^a \tag{5}$$

where dot and prime, respectively, represent derivatives with respect to temporal and radial coordinates. The model given above may evolve in any arbitrary way. To have a physically plausible model, we require that it must meet the pressure isotropy condition given by

$$T_1^1 - \tfrac{1}{2}\delta_1^1 T = T_2^2 - \tfrac{1}{2}\delta_2^2 T \tag{6}$$

yield a second-order partial differential equation

$$[A''/A + B''/B - (2B'/B + 1/r)(A'/A + B'/B)](1-kr^2/R^2) + (\dot{B}/B)kr/R^2 = 0 \tag{7}$$

We present solutions of the above equation relevant only for the cosmological context. Thus, we assume $A=1$ in equation (1). This reduces equation (7) to an equation in partial derivatives of B only. This equation can then be easily solved to obtain

$$B^{-1} = a\alpha(t)[1+\gamma(t)/a] \tag{8}$$

where $a=(1-kr^2/R^2)^{3/2}R^2/3k$, $\gamma(t)=\beta(t)/\alpha(t)$, and $\alpha(t)$ and $\beta(t)$ are integration constants. Note that for $K=0$ the solutions represented by equation (8) do not remain valid. Thus, in that case the solutions are sought from the beginning to give

$$B^{-1} = \beta(t)[1-\gamma(t)r^2] \tag{9}$$

where $\gamma(t)=\alpha(t)/\beta(t)$.

Corresponding to equation (8) and $A=1$, the expression for heat flow [assuming $\beta=$const. in equation (8)] becomes

$$q^a = \alpha/8\pi[2r(1-kr^2/R^2)^{3/2}](\partial\alpha/\partial t)\delta_1^a \tag{10}$$

which is the same condition as given in (1). The energy density and pressure, respectively, for these models are

$$\rho = (1/8\pi)B^{-2}\{3(\dot{B})^2 - B^{-2}[2BB'' - (B')^2 + 4BB'](1-kr^2/R^2) + (5B'/2B)kr/R^2\} \tag{11}$$

and

$$p = ((1/8\pi)1/B^2)\{(1-kr^2/R^2)B'/B^2 + [8B' - (3B'r + 4B)kr/R^2]1/2Br - 2(\dot{B}/B)^{\cdot} - 3(\dot{B}/B)^2\} \tag{12}$$

The conservation laws are identically satisfied. These laws corresponding to $T^{1a}{}_{;a}=0$ yield

$$\partial(B^2 q^a \delta_1^a)/\partial t + \partial[(1-kr^2/R^2)p+\rho]/\partial r = 0 \tag{13}$$

which defines a phenomenological temperature. There also arises an interesting equation defining the radial rate of change of ρ by

$$\partial\rho/\partial r = 3/2\ \partial B^2/\partial t\ q^a \delta_1^a + kr/R^2 \tag{14}$$

The other conservation equation corresponding to $T^{0a}{}_{;a}=0$ yields

$$\partial[(-g)^{1/2}\rho]/\partial t + \partial[(-g)^{1/2} q^a \delta_1^a]/\partial r + p\ \partial[(-g)^{1/2}]/\partial t = 0 \tag{15}$$

which again remains unchanged as in (1). As in Bokhari (n.d.), the method used to derive a hyperbolic equation of heat conduction is not applicable here either since the term $q_a u^b u^a{}_{;b} + u^a u^b q_{a;b}$ vanishes identically due to the form of the metric we have used.

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